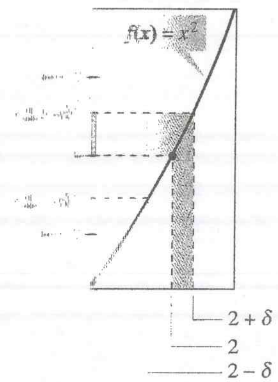


The limit of f(x) as x approaches 2 is 4.
Figure 1.15

EXAMPLE 8 Using the ε-δ Definition of a Limit

Use the ε-δ definition of a limit to prove that

$$\lim_{x\to 2} x^2 = 4.$$

Solution You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon \text{ when } 0 < |x - 2| < \delta.$$

To find an appropriate δ, begin by writing $|x^2 - 4| = |x - 2||x + 2|$. For all x in the interval (1, 3), you know that $|x + 2| < 5$. Thus, letting δ be the minimum of $\varepsilon/5$ and 1, it follows that, whenever $0 < |x - 2| < \delta$, you have

$$|x^2 - 4| = |x - 2||x + 2| < \left(\frac{\varepsilon}{5}\right)(5) = \varepsilon$$

as shown in Figure 1.15.

Throughout this chapter you will use the ε-δ definition of a limit primarily to prove theorems about limits and to establish the existence or nonexistence of particular types of limits. For *finding* limits, you will learn techniques that are easier to use than the ε-δ definition of a limit.

EXERCISES FOR SECTION 1.2

1. ***Modeling Data*** The cost of a telephone call between two cities is \$0.75 for the first minute and \$0.50 for each additional minute. A formula for the cost is given by

$$C(t) = 0.75 - 0.50[\![-(t - 1)]\!]$$

where t is the time in minutes.

(Note: $[\![x]\!]$ = greatest integer n such that $n \le x$. For example, $[\![3.2]\!] = 3$ and $[\![-1.6]\!] = -2$.)

(a) Use a graphing utility to graph the cost function for $0 < t \le 5$.

(b) Use the graph to complete the table and observe the behavior of the function as t approaches 3.5. Use the graph and the table to find

$$\lim_{t\to 3.5} C(t).$$

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

(c) Use the graph to complete the table and observe the behavior of the function as t approaches 3.

t	2	2.5	2.9	3	3.1	3.5	4
C				?			

Does the limit of $C(t)$ as t approaches 3 exist? Explain.

2. ***Writing*** Write a brief description of the meaning of the notation

$$\lim_{x\to 8} f(x) = 25.$$

In Exercises 3–10, complete the table and use the result to estimate the limit. If desired, use a graphing utility to graph the function to confirm your result.

3. $\displaystyle\lim_{x\to 2} \frac{x - 2}{x^2 - x - 2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

4. $\displaystyle\lim_{x\to 2} \frac{x - 2}{x^2 - 4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

5. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x}$

x	−0.1	−0.01	−0.001	0.001	0.01	1.0
$f(x)$						

6. $\lim_{x \to -3} \dfrac{\sqrt{1-x}-2}{x+3}$

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$						

7. $\lim_{x \to 3} \dfrac{[1/(x+1)]-(1/4)}{x-3}$

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$						

8. $\lim_{x \to 4} \dfrac{[x/(x+1)]-(4/5)}{x-4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

9. $\lim_{x \to 0} \dfrac{\sin x}{x}$

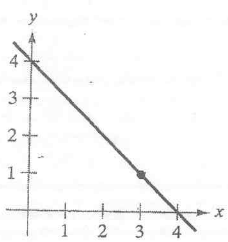
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

10. $\lim_{x \to 0} \dfrac{\cos x - 1}{x}$

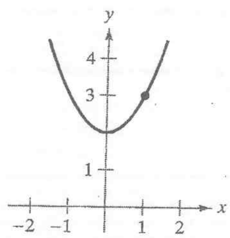
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

In Exercises 11–20, use the graph to find the limit (if it exists).

11. $\lim_{x \to 3} (4-x)$

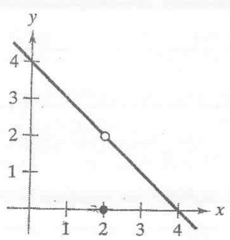


12. $\lim_{x \to 1} (x^2+2)$



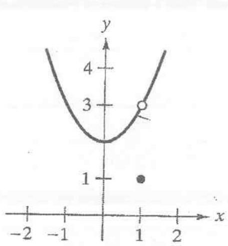
13. $\lim_{x \to 2} f(x)$

$$f(x) = \begin{cases} 4-x, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

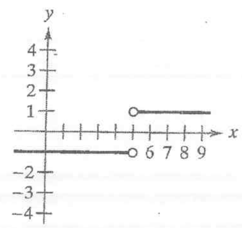


14. $\lim_{x \to 1} f(x)$

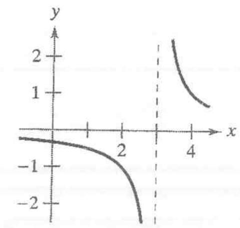
$$f(x) = \begin{cases} x^2+2, & x \neq 1 \\ 1, & x = 1 \end{cases}$$



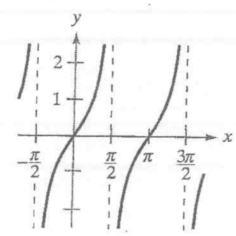
15. $\lim_{x \to 5} \dfrac{|x-5|}{x-5}$



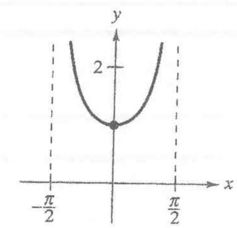
16. $\lim_{x \to 3} \dfrac{1}{x-3}$



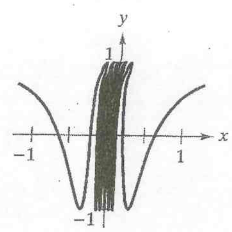
17. $\lim_{x \to \pi/2} \tan x$



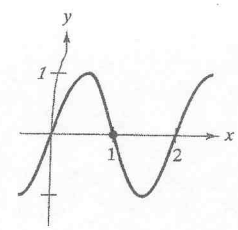
18. $\lim_{x \to 0} \sec x$



19. $\lim_{x \to 0} \cos \dfrac{1}{x}$



20. $\lim_{x \to 1} \sin \pi x$



In Exercises 21–24, find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

21. $\lim_{x \to 2} (3x+2)$

22. $\lim_{x \to 4} \left(4 - \dfrac{x}{2}\right)$

23. $\lim_{x \to 2} (x^2-3)$

24. $\lim_{x \to 5} (x^2+4)$

In Exercises 25–36, find the limit L. Then use the ε-δ definition to prove that the limit is L.

25. $\lim_{x \to 2} (x+3)$

26. $\lim_{x \to -3} (2x+5)$

27. $\lim_{x \to -4} \left(\frac{1}{2}x - 1\right)$

28. $\lim_{x \to 1} \left(\frac{2}{3}x + 9\right)$

29. $\lim_{x \to 6} 3$

30. $\lim_{x \to 2} (-1)$

31. $\lim_{x \to 0} \sqrt[3]{x}$

32. $\lim_{x \to 4} \sqrt{x}$

33. $\lim_{x \to -2} |x-2|$

34. $\lim_{x \to 3} |x-3|$

35. $\lim_{x \to 1} (x^2+1)$

36. $\lim_{x \to -3} (x^2+3x)$